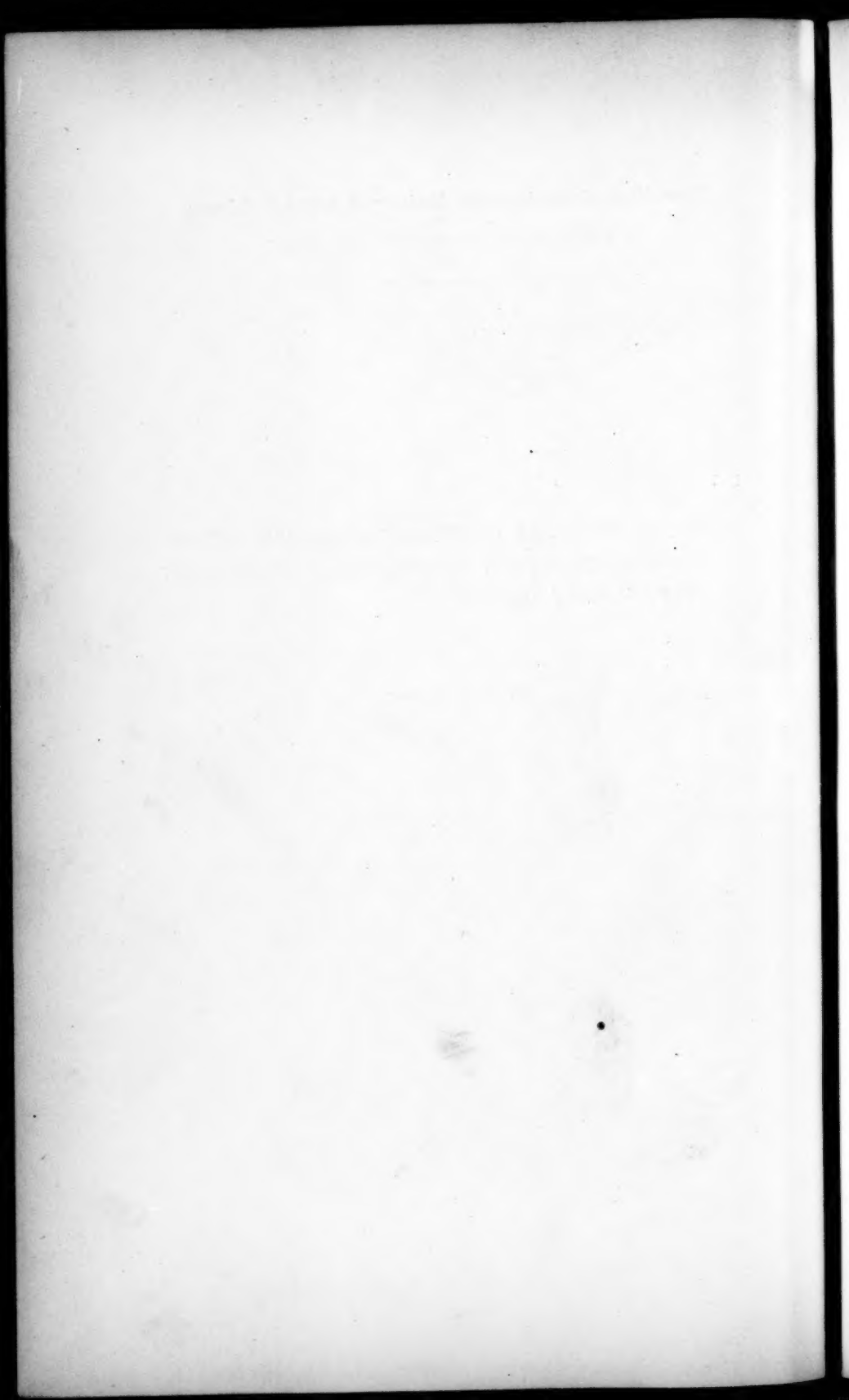


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BY OTTO DUNKEL.



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WE will consider in the present paper a system of n differential equations of the form :

$$(1) \quad \frac{dy_i}{dx} = \sum_{j=1}^{j=n} \left(\frac{\mu_{i,j}}{x} + a_{i,j} \right) y_j \quad (i = 1, 2, \dots, n),$$

in which the $\mu_{i,j}$'s are constants, and the $a_{i,j}$'s are functions, not necessarily analytic, of the real independent variable x , continuous in the interval :

$$0 < x \leq b.†$$

We shall require that $|a_{i,j}|$ be integrable up to the point $x = 0$. For the development of certain sets of solutions we shall make the additional requirement that even after certain powers of $\log x$ have been multiplied into $|a_{i,j}|$ the resulting function shall be integrable up to the point $x = 0$; this requirement will be stated more explicitly later.‡

The point $x = 0$ satisfying these conditions may be called a *regular singular point* of the system of equations (1) in conformity with the use of that term by Professor Bôcher in the study of linear differential

* This paper was accepted in June, 1902, by the Faculty of Arts and Sciences of Harvard University in fulfilment of the requirement of a thesis for the degree of Doctor of Philosophy.

† The requirement that the functions $a_{i,j}$ should be continuous in $0 < x \leq b$ is made only for the sake of simplicity. We might allow them to have a finite number of discontinuities in $0b$ of such a kind that each function $|a_{i,j}|$ can be integrated throughout the interval; and all the following work would hold with very little change.

‡ Cf. p. 367.

equations of the second order.* That this terminology is a legitimate extension of that commonly used when the coefficients of the system of differential equations are analytic functions of a complex variable, will be evident if the results of the present paper are compared with the thesis by Sauvage: *Théorie générale des systèmes d'équations différentielles linéaires et homogènes*.†

Our object is to investigate the nature of the solutions of (1) in the neighborhood of the regular singular point $x = 0$; and for this purpose we shall first reduce the system of equations to a canonical form by means of a linear transformation with constant coefficients of the dependent variable. We shall then apply the method of successive approximations to develop about the point $x = 0$ a system of n linearly independent solutions of the canonical system. By means of the linear transformation we shall return to n linearly independent solutions of the original system; and finally an application will be made to the case of the single homogeneous linear differential equation of the n th order.

§ 1.

A SPECIAL SYSTEM OF EQUATIONS: ITS REDUCTION TO A CANONICAL FORM, AND SOLUTION.‡

Let us first examine the special case of (1) in which the coefficients $a_{i,j}$ are all zero. In this case we have the system of differential equations:

$$(3) \quad \frac{dy_i}{dx} = \sum_{j=1}^{j=n} \frac{\mu_{i,j}}{x} y_j \quad (i = 1, 2, \dots, n),$$

where the $\mu_{i,j}$'s are constants.

A solution of this system may be obtained in the following way. Substitute

$$y_i = C_i x^{C_i} \quad C_i = \text{constant}$$

* Cf. Trans. Am. Math. Soc., Vol. I. Jan. 1900, p. 41. The results of this paper are included as a special case in those we now give. Cf. § 7.

† Paris, 1895. Reprinted from the *Annales de la Faculté des Sciences de Toulouse*, Vols. VIII. and IX.

‡ The results of this section are not new, being on the one hand only slightly modified forms of Weierstrass's results (cf. the foot-note on p. 345), and on the other hand special cases of the results obtained by Sauvage (cf. the last foot-note.)

in the equations (3), and then determine the constants r and C_i so that the equations are satisfied. We obtain in this way the following system of n homogeneous linear equations for the C 's:

$$(4) \quad \mu_{i,1} C_1 + \dots + (\mu_{i,i} - r) C_i + \dots + \mu_{i,n} C_n = 0 \quad (i = 1, 2, \dots, n).$$

The necessary and sufficient condition that (4) may be satisfied by a set of C 's not all zero is that the determinant:

$$(5) \quad \Delta(r) = \begin{vmatrix} \mu_{1,1} - r & . & . & . & \mu_{1,n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \mu_{n,1} & . & . & . & \mu_{n,n} - r \end{vmatrix}$$

shall vanish. This determinant equated to zero gives an equation of the n th degree in r , which is called the *characteristic equation* of (3);* the determinant itself we may call the *characteristic determinant* of (3).

If the characteristic equation has n distinct roots r_1, r_2, \dots, r_n , we can determine n linearly independent solutions of (3) of the form:

$$(6) \quad y_{i,j} = C_{i,j} x^{r_j} \quad \begin{pmatrix} i = 1, 2, \dots, n \\ j = 1, 2, \dots, n \end{pmatrix}.$$

If however there is a multiple root, there will be, in general, solutions involving powers of $\log x$. To determine these solutions, we must examine the minors of the characteristic determinant (5), and ascertain if this multiple root is also a root of all the first minors, second minors, etc. For the further study of this case, it will be useful to introduce the conception of the *elementary divisors* of $\Delta(r)$.

Suppose r' is a root of the determinant $\Delta(r)$ such that all the p th minors of $\Delta(r)$ are divisible by $(r - r')^{l_p}$, but no higher power of $r - r'$ divides them all. In the same way $(r - r')^{l_p+1}$ shall be the highest power of $r - r'$ dividing all the $(p + 1)$ th minors. Then the expression:

$$(r - r')^{e_p} \quad e_p = l_p - l_{p+1}$$

* Sauvage: *l. c.* p. 80.

is called by Weierstrass an elementary divisor of the determinant $\Delta(r)$.^{*} It will be convenient to employ a different notation from that used in the definition of an elementary divisor. An elementary divisor of $\Delta(r)$ will be written:

$$(r - r_k)^{e_k},$$

and it is to be noticed that several r 's with different subscripts, may be equal, as will be the case when a multiple root furnishes several elementary divisors. We shall always have:

$$(7) \quad \Delta(r) = \prod_{k=1}^{k=m} (r_k - r)^{e_k} \quad \sum_{k=1}^{k=m} e_k = n.$$

It can be shown (cf. the next foot-note) that a necessary and sufficient condition that a pair of systems of differential equations:

$$\frac{dy_i}{dx} = \sum_{j=1}^{j=n} \frac{\mu_{i,j}}{x} y_j, \quad \frac{dz_i}{dx} = \sum_{j=1}^{j=n} \frac{\nu_{i,j}}{x} z_j$$

$$(i = 1, 2, \dots, n)$$

can be transformed the one into the other by means of a transformation:

$$y_i = \sum_{j=1}^{j=n} A_{i,j} z_j \quad (i = 1, 2, \dots, n),$$

in which the A 's are constants whose determinant is not zero, is that the characteristic determinants of the two systems have the same elementary divisors. This theorem enables us to simplify the solution of the system (3); for we can write down a second system of differential equations having the same elementary divisors as (3), as follows:

$$(8) \quad \frac{dz_{k,l}}{dx} = \frac{1}{x} z_{k,l-1} + \frac{r_k}{x} z_{k,l} \quad \left(\begin{matrix} k = 1, 2, \dots, m \\ l = 1, 2, \dots, e_k \end{matrix} \right),$$

where

$$z_{k,0} = 0.$$

* Cf. Muth: Theorie und Anwendung der Elementartheiler, p. 2

The characteristic determinant of (8) is :

$$(9) \Delta(r) = \begin{vmatrix} \begin{matrix} r_1 - r & & & \\ 1 & r_1 - r & & \\ & \ddots & \ddots & \\ & & 1 & r_1 - r \end{matrix} & & \\ & \begin{matrix} r_2 - r & & & \\ 1 & r_2 - r & & \\ & \ddots & \ddots & \\ & & 1 & r_2 - r \end{matrix} & & \\ & & \begin{matrix} r_m - r & & & \\ 1 & r_m - r & & \\ & \ddots & \ddots & \\ & & 1 & r_m - r \end{matrix} & & \end{vmatrix}$$

and it will be easily seen that it has the elementary divisors $(r - r_1)^{e_1}$, $(r - r_2)^{e_2}$, . . . $(r - r_m)^{e_m}$. Then, by the theorem above referred to, there exists a set of n^2 constants $A_{i,k,l}$, whose determinant is not zero, such that :

$$(10) \quad y_i = \sum_{k=1}^{k=m} \sum_{l=1}^{l=e_k} A_{i,k,l} z_{k,l}^* \quad (i = 1, 2, \dots, n).$$

The system of differential equations (8) we may speak of as the *canonical system* ; and now it is easily seen that :

The canonical system of equations admits e_κ solutions, corresponding to the elementary divisor $(r - r_\kappa)^{e_\kappa}$, of the following simple form :

$$(11) \quad \begin{aligned} z_{k,l} &= 0 & k &\neq \kappa \\ z_{\kappa,l} &= 0 & l &< \lambda \\ z_{\kappa,l} &= \frac{1}{(l-\lambda)!} x^{r_\kappa} (\log x)^{l-\lambda} & \lambda &\leq l \leq e_\kappa \end{aligned} \quad (\lambda = 1, 2, \dots, e_\kappa)$$

and the n solutions obtained by giving κ the values $1, 2, \dots, m$ are linearly independent.

* Cf. Weierstrass, Werke, Vol. II. pp. 75, 76. The case considered by Weierstrass is very easily reduced to the one we are considering by the change of independent variable $t = \log x$. This reduction of Weierstrass is also given in Muth's *Elementartheiler*, pp. 195, 198. On page 198 are a number of references to the use of the theory of elementary divisors in the study of differential equations.

The determinant of these n solutions can be written out in such a way that the elements above the principal diagonal are all zero. Thus the value of the determinant is:

$$x^{\sum_{\kappa=1}^{n-m} e_{\kappa} r_{\kappa}},$$

and the n solutions are therefore linearly independent.

On account of the relation (10), each solution (11) of the system (8) will determine a solution of the original system (3). Accordingly we have n solutions of (3) which are linearly independent, for their determinant at any point is equal to the determinant of the A 's in (10) multiplied into the determinant of the solutions (11) for the same point, and neither of these determinants is zero.

Suppose now we consider any multiple root of the characteristic determinant (5); for simplicity let us take r_1 , and suppose that

$$(12) \quad r_1 = r_2 = \dots = r_k,$$

so that

$$(13) \quad e_1 + e_2 + \dots + e_k$$

is the multiplicity of the root r_1 . Then from (10) and (11) we see that, corresponding to this root, there are k solutions of (3) not involving $\log x$ and linearly independent:

$$(14) \quad y_i^{e_i, e_k} = A_{i, \kappa, e_k} x^{r_1} \quad \left(\begin{array}{l} i = 1, 2, \dots, n \\ \kappa = 1, 2, \dots, k \end{array} \right).$$

Therefore the constants A_{i, κ, e_k} are linearly independent solutions of the equations (4) when $r = r_1$, as we readily see by putting the values of $y_i^{e_i, e_k}$ in (3). Now the equations (4), in this case, have only k linearly independent solutions, since there are only k elementary divisors corresponding to r_1 , and therefore at least one k th minor in (5) is not zero when $r = r_1$. If, then, we have any other solution of (3) of the form:

$$y_i = C_i x^{r_1} \quad (i = 1, 2, \dots, n),$$

the constants C_i must be linear combinations of the k sets of constants A_{i, κ, e_k} . We have then the following result:

The system of equations (3) admits n linearly independent solutions, such that, corresponding to each elementary divisor $(r - r_k)^{e_k}$ of the characteristic determinant, there are e_k solutions:

$$(15) \quad y_i^{k,\lambda} = x^{r_k} \sum_{l=\lambda}^{l=e_k} \frac{1}{(l-\lambda)!} A_{i,k,l} (\log x)^{l-\lambda} \quad \left(\begin{matrix} i = 1, 2, \dots, n \\ \lambda = 1, 2, \dots, e_k \end{matrix} \right).$$

If r_k is a multiple root of the characteristic equation which furnishes s elementary divisors with the exponents $e_k, e_{k+1}, \dots, e_{k+s-1}$, then the constants:

$$A_{i,k,e_k}, A_{i,k+1,e_{k+1}}, \dots, A_{i,k+s-1,e_{k+s-1}}, \quad (i = 1, 2, \dots, n)$$

are s linearly independent solutions of the equations (4) when $r = r_k$.

§ 2.

SOLUTION OF THE CANONICAL SYSTEM IN THE GENERAL CASE BY SUCCESSIVE APPROXIMATIONS.

We shall now return to the system of equations (1); and here again we shall make use of the linear transformation (10) to reduce the system to the canonical form:

$$(16) \quad \frac{d}{dx} z_{k,l} = \frac{1}{x} z_{k,l-1} + \frac{r_k}{x} z_{k,l} + \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j} \quad * \\ (k = 1, 2, \dots, m) \quad (l = 1, 2, \dots, e_k).$$

The coefficients $b_{k,i}^{i,j}$ are linear functions with constant coefficients of the coefficients $a_{i,j}$ in (1).

We shall now make use of the method of successive approximations to develop solutions of (16) about the point $x = 0$. It will be convenient to write the equations (16) in the form:

$$(17) \quad \frac{d}{dx} z_{k,l} - \frac{1}{x} z_{k,l-1} - \frac{r_k}{x} z_{k,l} = \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j}.$$

The first approximation will be indicated by a third subscript 0, and is obtained as a solution of the system of equations resulting from (17) by making the right side zero:

* This reduction is used by Sauvage in the case of a system of equations with analytic coefficients. L. c., pp. 89, 90.

$$(18) \quad \frac{d}{dx} z_{k,l,0} - \frac{1}{x} z_{k,l-1,0} - \frac{r_k}{x} z_{k,l,0} = 0.$$

We have seen how to solve this system. Any one of the n solutions that we have obtained may be used as the first approximation. This approximation having been chosen, we insert it in the right side of (17) and obtain the following relations for the first correction:

$$\frac{d}{dx} z_{k,l,1} - \frac{1}{x} z_{k,l-1,1} - \frac{r_k}{x} z_{k,l,1} = \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j,0}.$$

The right side is now a known function of x ; and we have, consequently, a system of non-homogeneous linear differential equations to solve for $z_{k,l,1}$. Having determined this first correction, it is inserted in the right side of (17), and the resulting equations are solved for the second correction. This process is repeated again and again, the relation connecting the q th and the $(q+1)$ th correction being:

$$(19) \quad \frac{d}{dx} z_{k,l,q+1} - \frac{1}{x} z_{k,l-1,q+1} - \frac{r_k}{x} z_{k,l,q+1} = \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j,q}.$$

Each equation (19) may be written:

$$x^{r_k} \frac{d}{dx} (x^{-r_k} z_{k,l,q+1}) - \frac{1}{x} z_{k,l-1,q+1} = \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j,q},$$

whence:

$$(20) \quad z_{k,l,q+1} = x^{r_k} \left[\int_{c_{k,l}}^x x^{-1-r_k} z_{k,l-1,q+1} dx + \int_{c_{k,l}}^x x^{-r_k} \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j,q} dx \right].$$

Now writing out the value of $z_{k,l-1,q+1}$ in the same way, and substituting it in the first integral of (20), we have:

$$\begin{aligned} z_{k,l,q+1} = & x^{r_k} \left[\int_{c_{k,l}}^x \frac{1}{x} dx \int_{c_{k,l-1}}^x x^{-1-r_k} z_{k,l-2,q+1} dx \right. \\ & \left. + \int_{c_{k,l}}^x \frac{1}{x} dx \int_{c_{k,l-1}}^x x^{-r_k} \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j,q} dx + \int_{c_{k,l}}^x x^{-r_k} \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j,q} dx \right]. \end{aligned}$$

Now substitute in this result the value of $z_{k, l-2, q+1}$; in the result thus obtained the value of $z_{k, l-3, q+1}$, etc. After a certain number of substitutions we have:

$$(21) \quad z_{k, l, q+1} = x^{r_k} \left[\int_{c_{k, l}}^x \frac{1}{x} dx \dots \int_{c_{k, L+1}}^x x^{-1-r_k} z_{k, L, q+1} dx \right. \\ \left. + \sum_{t=1}^{l-L} \int_{c_{k, l}}^x \frac{1}{x} dx \dots \int_{c_{k, l+1-t}}^x x^{-r_k} \sum_{i=1}^{t-m} \sum_{j=1}^{e_{i, f}} b_{k, l+1-t}^{i, f} z_{i, j, q} dx \right],$$

where $L < l$.

When $L = 0$ we have:

$$(22) \quad z_{k, l, q+1} = x^{r_k} \sum_{t=1}^{l-1} \int_{c_{k, l}}^x \frac{1}{x} dx \dots \int_{c_{k, l+1-t}}^x x^{-r_k} \sum_{i=1}^{t-m} \sum_{j=1}^{e_{i, f}} b_{k, l+1-t}^{i, f} z_{i, j, q} dx.$$

The lower limits $c_{k, l}$ will be determined later to satisfy several conditions.

We may choose at pleasure any one of the m elementary divisors, say $(r - r_k)^{e_k}$, and then select any one of the corresponding e_k solutions of (18) for the first approximation. We shall take then for the first approximation $z_{k, l, 0}$, the values given in (11) for a particular λ ; the integers κ and λ will remain fixed for the solution we are now developing.

For the development of the solutions corresponding to $(r - r_k)^{e_k}$, we shall make the following further assumption as to the coefficients $b_{k, l}^{i, f}$. Let us examine all the exponents of the elementary divisors $(r - r_k)^{e_k}$, which are such that $Rr_k = Rr_\kappa$, where Rr_k means "real part of r_k ," and pick out one exponent, say e_κ , that is as great as any one in this special set of exponents; i. e.,

$$(23) \quad e_\kappa \geq e_k, \quad \text{where } Rr_k = Rr_\kappa.$$

The assumption is that the integrals:

$$\int_0^b |b_{k, l}^{i, f}| |\log x|^{e_\kappa-1} dx \quad \left(\begin{array}{l} k = 1, 2, \dots, m \\ l = 1, 2, \dots, e_k \end{array} \right)$$

converge. If in particular r_κ is a simple root, and no multiple root has the same real part as it, then $e_\kappa - 1 = 0$, and this further restriction drops out. Or it might happen that r_κ is a multiple root, but that all the

exponents of the set (23) are unity, and in this case the restriction drops out also.

The lower limits of integration in (22) will be determined as follows :

$$(24) \quad \left\{ \begin{array}{ll} \text{I. } Rr_k > Rr_\kappa & c_{k,l} = c \\ \text{II. } Rr_k = Rr_\kappa & \begin{cases} c_{k,l} = 0 & l \leq L = e_\kappa - e_\kappa + \lambda \\ c_{k,l} = c & l > L \end{cases} \\ \text{III. } Rr_k < Rr_\kappa & c_{k,l} = 0 \end{array} \right.$$

where c is a constant not zero, which will be determined more closely later (cf. p. 358). It will be proved in § 3, that even in the cases in which the lower limit is zero, the integrals converge.

When all these conditions have been satisfied, we build from the first approximation and the successive corrections the n infinite series :

$$(25) \quad z_{k,l} = \sum_{q=0}^{q=\infty} z_{k,l,q} \quad \left(\begin{array}{l} k = 1, 2, \dots m \\ l = 1, 2, \dots e_k \end{array} \right),$$

which will be proved in § 4 to converge and to form a solution of the system of equations (16).

It will be convenient to consider in place of the functions $z_{k,l,q}$ certain new functions $\phi_{k,l,q}$, which will be defined by the following formulae :

$$(26) \quad \begin{aligned} z_{k,l,q} &= x^{r_\kappa} \phi_{k,l,q} & \left\{ \begin{array}{l} \text{when } Rr_k \neq Rr_\kappa, \\ \text{or } Rr_k = Rr_\kappa, \ k \neq \kappa \text{ and } l \leq L, \\ \text{or } \qquad \qquad \qquad k = \kappa \text{ and } l \leq \lambda, \end{array} \right. \\ z_{k,l,q} &= x^{r_\kappa} (\log x)^{l-L} \phi_{k,l,q} & Rr_k = Rr_\kappa, \ k \neq \kappa \text{ and } L < l \leq e_k, \\ z_{k,l,q} &= x^{r_\kappa} (\log x)^{l-\lambda} \phi_{k,l,q} & \lambda < l \leq e_\kappa. \end{aligned}$$

For the case of $q = 0$, i. e., the first approximation, the ϕ 's are certain constants (cf. (11)); for all other values of q the formulae (26) define them as continuous functions of x so long as x is not zero; and we shall see later (Proof of Convergence, p. 358 et seq.) that each one approaches zero when x approaches zero. We shall therefore define each $\phi_{k,l,q}$, when $q > 0$, as zero for $x = 0$; and with this definition they will be continuous functions of x in the whole interval Ob .

These ϕ 's can be computed from the following recurrent formulae, which are easily obtained from (21), (22), (24), and (26) :

I. $Rr_k > Rr_\kappa$.

$$(27) \quad \phi_{k,l,q+1} = x^{r_k-r_\kappa} \sum_{i=1}^{l-1} \int_c^x \frac{1}{x} dx \dots$$

$$\dots \int_c^x \frac{1}{x} dx \int_c^x x^{-(r_k-r_\kappa)} \sum_{i=1}^{l-m} \sum_{j=1}^{e_i} b_{k,i+1-i}^{i,j} \phi_{i,j,q} (\log x)^{h_{i,j}} dx,$$

l integrations

where $h_{i,j}$ is a positive integer whose value it will not be necessary to write out, noting, however, that:

$$(28) \quad h_{i,j} \leq e_k - \lambda.$$

II. $Rr_k = Rr_\kappa$.a) $k \neq \kappa, \quad l \leq L$.

$$(29) \quad \phi_{k,l,q+1} = x^{r_k-r_\kappa} \sum_{i=1}^{l-1} \int_0^x \frac{1}{x} dx \dots$$

$$\dots \int_0^x \frac{1}{x} dx \int_0^x x^{-(r_k-r_\kappa)} \sum_{i=1}^{l-m} \sum_{j=1}^{e_i} b_{k,i+1-i}^{i,j} \phi_{i,j,q} (\log x)^{h_{i,j}} dx.$$

l integrations

b) $k \neq \kappa, \quad L < l \leq e_k$.

$$(30) \quad \phi_{k,l,q+1} = \frac{x^{r_k-r_\kappa}}{(\log x)^{l-L}} \left[\int_c^x \frac{1}{x} dx \dots \int_c^x \frac{1}{x} dx \int_c^x \frac{1}{x} x^{-(r_k-r_\kappa)} \phi_{k,L,q+1} dx \right.$$

$(l-L)$ integrations

$$\left. + \sum_{i=1}^{l-L} \int_c^x \frac{1}{x} dx \dots \int_c^x \frac{1}{x} dx \int_c^x x^{-(r_k-r_\kappa)} \sum_{i=1}^{l-m} \sum_{j=1}^{e_i} b_{k,i+1-i}^{i,j} \phi_{i,j,q} (\log x)^{h_{i,j}} dx \right].$$

l integrations

c) $k = \kappa, \quad l \leq \lambda$.

$$(31) \quad \phi_{\kappa,l,q+1} = \sum_{i=1}^{l-1} \int_0^x \frac{1}{x} dx \dots$$

$$\dots \int_0^x \frac{1}{x} dx \int_0^x \sum_{i=1}^{l-m} \sum_{j=1}^{e_i} b_{\kappa,i+1-i}^{i,j} \phi_{i,j,q} (\log x)^{h_{i,j}} dx.$$

l integrations

$$d) \quad k = \kappa, \quad \lambda < l \leq L.$$

$$(32) \quad \phi_{\kappa, l, \varphi+1} = \frac{1}{(\log x)^{l-\lambda}} \sum_{i=1}^{t=l} \int_0^x \frac{1}{x} dx \dots$$

$$\dots \int_0^x \frac{1}{x} dx \int_0^x \sum_{i=1}^{t=m} \sum_{j=1}^{j=e_i} b_{\kappa, l+1-i}^{i,j} \phi_{i,j,\varphi} (\log x)^{h_{i,j}} dx.$$

t integrations

$$e) \quad k = \kappa, \quad L < l \leq e_{\kappa}.$$

$$(33) \quad \phi_{\kappa, l, \varphi+1} = \frac{1}{(\log x)^{l-\lambda}} \left[\int_c^x \frac{1}{x} dx \dots \int_c^x \frac{1}{x} dx \int_c^x \frac{1}{x} \phi_{\kappa, L, \varphi+1} (\log x)^{e_{\kappa}-e_{\kappa}} dx \right.$$

$(l-L)$ integrations

$$\left. + \sum_{i=1}^{t=l-L} \int_c^x \frac{1}{x} dx \dots \int_c^x \frac{1}{x} dx \int_c^x \sum_{i=1}^{t=m} \sum_{j=1}^{j=e_i} b_{\kappa, l+1-i}^{i,j} \phi_{i,j,\varphi} (\log x)^{h_{i,j}} dx \right].$$

t integrations

III. $R r_{\kappa} < R r_{\kappa}$.

Here we have the formula (29) again.

In § 4 we shall consider the n series :

$$(34) \quad \phi_{k,l} = \sum_{q=0}^{q=\infty} \phi_{k,l,q} \quad \left(\begin{matrix} k = 1, 2, \dots m \\ l = 1, 2, \dots e_k \end{matrix} \right),$$

which are such that if we multiply each by its proper factor $x^{r_{\kappa}} (\log x)^{h_{k,l}}$, where $h_{k,l}$ is given in (26), we obtain the n series (25). We thus reduce the proof of convergence of (25) to the question of the convergence of (34), and this last question will be settled by reference to certain formulae to be established in the next section.

§ 3.

LEMMAS CONCERNING MULTIPLE INTEGRALS.

We now prove a number of lemmas, which will be useful in the proof of convergence, and which also verify the statement that we have made that the integrals of the last section, in which the lower limit is zero, converge.

LEMMA I. If b is a function of x , continuous in the interval $0 < x \leq c$, and such that $|b| |\log x|^{t-1}$, (t an integer ≥ 1) is in-

tegrable up to $x = 0$, then b can be integrated t times from $x = 0$ as follows:

$$(35) \quad f_t(x) = \int_0^x \frac{1}{x} dx \dots \int_0^x \frac{1}{x} dx \int_0^x b dx.$$

integrations

LEMMA II. If the conditions of Lemma I. hold, then:

$$(36) \quad |f_t(x)| = \left| \int_0^x \frac{1}{x} dx \dots \int_0^x \frac{1}{x} dx \int_0^x b dx \right| \leq \int_0^x |b| |\log x|^{t-1} dx.$$

integrations

($0 < x \leq 1$).

We will prove these lemmas by mathematical induction. They are true when $t = 1$. Let us assume that they are true for a particular value of t , say $t = t_1$.

Let X be any particular value of x in the interval $0 < x \leq 1$, and choose ϵ at pleasure such that $0 < \epsilon < X$. Then we have:

$$\begin{aligned} \left| \int_0^X \frac{1}{x} f_{t_1}(x) dx \right| &\leq \int_0^X \frac{1}{x} |f_{t_1}(x)| dx \leq \int_0^X \frac{1}{x} dx \int_0^x |b| |\log x|^{t_1-1} dx \\ &= \log X \int_0^X |b| |\log x|^{t_1-1} dx - \log \epsilon \int_0^\epsilon |b| |\log x|^{t_1-1} dx \\ &\quad - \int_\epsilon^X \log x |b| |\log x|^{t_1-1} dx \\ &= |\log \epsilon| \int_0^\epsilon |b| |\log x|^{t_1-1} dx + \int_\epsilon^X |b| |\log x|^{t_1} dx \\ &\quad - |\log X| \int_0^X |b| |\log x|^{t_1-1} dx \\ &\leq |\log \epsilon| \int_0^\epsilon |b| |\log x|^{t_1-1} dx + \int_\epsilon^X |b| |\log x|^{t_1} dx \\ &\leq \int_0^\epsilon |b| |\log x|^{t_1} dx + \int_\epsilon^X |b| |\log x|^{t_1} dx = \int_0^X |b| |\log x|^{t_1} dx. \end{aligned}$$

Therefore, when ϵ approaches zero, the integral on the left side of all these inequalities converges, and we have:

$$|f_{t_1+1}(X)| = \left| \int_0^X \frac{1}{x} f_{t_1}(x) dx \right| \leq \int_0^X |b| |\log x|^{t_1} dx.$$

Thus the two lemmas are proved when $x \leq 1$; and it is easily seen how to conclude the proof of I. in case $c > 1$.

LEMMA III. If b is a continuous function of x in the interval $0 < x \leq c$, and its absolute value is integrable up to $x = 0$, and if:

$$F_t(x) = \int_0^x \frac{1}{x} dx \dots \int_0^x \frac{1}{x} dx \int_0^x x^r b dx$$

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where r is real and greater than zero, then:

$$(37) \quad |x^{-r} F_t(x)| \leq \frac{1}{r^{t-1}} \int_0^x |b| dx.$$

When $t = 1$ we have:

$$|x^{-r} F_1(x)| = x^{-r} \left| \int_0^x x^r b dx \right| \leq x^{-r} \int_0^x x^r |b| dx \leq \int_0^x |b| dx,$$

and so in this case III. is true. Assume that it is true for $t = t_1$; then it is also true for $t = t_1 + 1$. For:

$$\begin{aligned} |x^{-r} F_{t_1+1}(x)| &= x^{-r} \left| \int_0^x \frac{1}{x} F_{t_1}(x) dx \right| \leq x^{-r} \int_0^x \frac{1}{x} |F_{t_1}(x)| dx \\ &\leq x^{-r} \int_0^x \frac{1}{x} \left[\frac{x^r}{r^{t_1-1}} \int_0^x |b| dx \right] dx \\ &\leq \frac{x^{-r}}{r^{t_1-1}} \left[\int_0^x x^{-1+r} dx \right] \left[\int_0^x |b| dx \right] = \frac{1}{r^{t_1}} \int_0^x |b| dx. \end{aligned}$$

Therefore III. is true for all values of t .

LEMMA IV. If b is a continuous function of x in the interval $0 < x \leq c$, and if:

$$g_t(x) = \int_c^x \frac{1}{x} dx \dots \int_c^x \frac{1}{x} dx \int_c^x x^{-r} b dx$$

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where r is real and greater than zero, then:

$$(38) \quad |x^r g_t(x)| \leq \frac{1}{r^{t-1}} \int_c^x |b| |dx| \quad (0 < x \leq c).$$

When $t = 1$ we have:

$$|x^r g_1(x)| \leq x^r \int_c^x x^{-r} |b| |dx| \leq \int_c^x |b| |dx|.$$

Assume that IV. is true for $t = t_1$. Then:

$$\begin{aligned} |x^r g_{t_1+1}(x)| &\leq x^r \int_c^x \frac{1}{x} |g_{t_1}(x)| |dx| \leq \frac{x^r}{r^{t_1-1}} \int_c^x x^{-1-r} \left[\int_c^x |b| |dx| \right] |dx| \\ &\leq \frac{x^r}{r^{t_1-1}} \left[\int_c^x x^{-1-r} |dx| \right] \left[\int_c^x |b| |dx| \right] = \frac{1}{r^{t_1}} \left[1 - \left(\frac{x}{c} \right)^r \right] \int_c^x |b| |dx| \\ &\leq \frac{1}{r^{t_1}} \int_c^x |b| |dx|. \end{aligned}$$

Therefore IV. is true for $t = t_1 + 1$, and the lemma is proved.

LEMMA V. If b , r , and g , are defined as in IV. and it is further assumed that the absolute value of b is integrable up to $x = 0$, then:

$$(39) \quad \lim_{x \rightarrow 0} x^r g_t(x) = 0.$$

To prove this, let us choose a constant s such that $0 < s < r$. Then:

$$\begin{aligned} |x^r g_t(x)| &= x^r \left| x^{-r} \int_c^x \frac{1}{x} dx \dots \int_c^x \frac{1}{x} dx \int_c^x x^{-(r-s)} x^{-s} b dx \right| \\ &\leq \frac{x^s}{(r-s)^{t-1}} \int_c^x x^{-s} |b| |dx|, \end{aligned}$$

as we see from IV. by replacing r by $r - s$ and b by $x^{-s} b$.

If ϵ is a positive number chosen arbitrarily small, we can choose η so near 0 that :

$$\int_0^{\eta} |b| |dx| \leq \frac{\epsilon}{2},$$

and then $X \leq \eta$ so that :

$$X^s \int_c^{\eta} x^{-s} |b| |dx| \leq \frac{\epsilon}{2}.$$

Then :

$$\begin{aligned} x^s \int_c^x x^{-s} |b| |dx| &= x^s \int_c^{\eta} x^{-s} |b| |dx| + x^s \int_{\eta}^x x^{-s} |b| |dx| \\ &\leq X^s \int_c^{\eta} x^{-s} |b| |dx| + \int_{\eta}^x |b| |dx| \quad (0 < x \leq X \leq \eta) \\ &\leq \frac{\epsilon}{2} + \int_{\eta}^0 |b| |dx| \leq \epsilon. \end{aligned}$$

Therefore :

$$|x^s g_s(x)| \leq \frac{\epsilon}{(r-s)^{s-1}} \quad (0 < x \leq X)$$

and V. is proved.

LEMMA VI. If β is a function of x continuous in the interval $0 < x \leq c$, and

$$(40) \quad \lim_{x \rightarrow 0} \beta = 0;$$

and if :

$$G_t(x) = \int_c^x \frac{1}{x} dx \dots \int_c^x \frac{1}{x} \beta dx$$

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then :

$$(41) \quad \lim_{x \rightarrow 0} \frac{1}{(\log x)^t} G_t(x) = 0.$$

When $t = 0$ it is obvious that (41) is true, for then :

$$\lim_{x \rightarrow 0} G_0(x) = \lim_{x \rightarrow 0} \beta = 0.$$

Assume now that (41) is true when $t = t_1$; then it will also be true for $t = t_1 + 1$. For if ϵ is a positive number chosen arbitrarily small, we can choose η so near 0 that :

$$\left| \frac{1}{(\log x)^{t_1}} G_{t_1}(x) \right| \leq \frac{\epsilon}{2} (t_1 + 1) \quad (0 < x \leq \eta < 1)$$

$$\begin{aligned} G_{t_1+1}(x) &= \int_c^x \frac{1}{x} G_{t_1}(x) dx = \int_c^\eta \frac{1}{x} G_{t_1}(x) dx + \int_\eta^x \frac{1}{x} G_{t_1}(x) dx \\ \left| \int_\eta^x \frac{1}{x} G_{t_1}(x) dx \right| &\leq \int_\eta^x \frac{1}{x} |G_{t_1}(x)| |dx| \leq \frac{\epsilon}{2} (t_1 + 1) \int_\eta^x \frac{|\log x|^{t_1}}{x} |dx| \\ &\quad (0 < x \leq \eta) \\ &\leq \frac{\epsilon}{2} (t_1 + 1) \left| \frac{(\log x)^{t_1+1} - (\log \eta)^{t_1+1}}{t_1 + 1} \right| \\ &= \frac{\epsilon}{2} |\log x|^{t_1+1} \left[1 - \left(\frac{\log \eta}{\log x} \right)^{t_1+1} \right] \leq \frac{\epsilon}{2} |\log x|^{t_1+1}. \end{aligned}$$

Now choose X in the interval $0 < x \leq \eta$ so that:

$$\left| \frac{1}{(\log x)^{t_1+1}} \int_c^\eta \frac{1}{x} G_{t_1}(x) dx \right| \leq \frac{\epsilon}{2} \quad (0 < x \leq X).$$

Then:

$$\begin{aligned} \left| \frac{1}{(\log x)^{t_1+1}} G_{t_1+1}(x) \right| &\leq \left| \frac{1}{(\log x)^{t_1+1}} \int_c^\eta \frac{1}{x} G_{t_1}(x) dx \right| + \left| \frac{1}{(\log x)^{t_1+1}} \int_\eta^x \frac{1}{x} G_{t_1}(x) dx \right| \leq \epsilon, \\ &\quad (0 < x \leq X), \end{aligned}$$

and:

$$\lim_{x \rightarrow 0} \frac{1}{(\log x)^{t_1+1}} G_{t_1+1}(x) = 0.$$

Therefore VI. is true for all values of t .

LEMMA VII. If β is a continuous function of x in the interval $0 < x \leq c \leq 1$, and such that it is not greater in absolute value than the constant N , and if G_t is defined as in VI, then:

$$(42) \quad |G_t(x)| = \left| \int_c^x \frac{1}{x} dx \dots \int_c^x \frac{1}{x} \beta dx \right| \leq N |\log x|^t.$$

For:

$$\begin{aligned} |G_t(x)| &\leq \int_c^x \frac{1}{x} |dx| \dots \int_c^x \frac{1}{x} N |dx| \leq N \int_c^x \frac{1}{x} |dx| \dots \int_c^x \frac{1}{x} |dx| \\ &\leq N \left[\int_c^x \frac{1}{x} |dx| \right]^t \leq N |\log x|^t \left[1 - \frac{\log c}{\log x} \right]^t \leq N |\log x|^t. \end{aligned}$$

§ 4.

PROOF OF CONVERGENCE.

The convergence of the series (34) will now be proved; and in this proof it will also be shown that:

$$(43) \quad \lim_{x \rightarrow 0} \phi_{k,i,q} = 0 \quad q > 0.$$

We shall use in the proof the following functions. A function of x , B , is chosen having the same properties of integrability as $b_{k,i}^{i,j}$, and such that:

$$(44) \quad B \geq \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} |b_{k,i}^{i,j}| * \quad \left(\begin{matrix} k=1, 2, \dots, m \\ l=1, 2, \dots, e_k \end{matrix} \right).$$

Next consider all the differences $Rr_k - Rr_\kappa$ which are not zero, and choose a positive constant d such that:

$$0 < d \leq |Rr_k - Rr_\kappa|.$$

Then U shall be a positive constant such that:

$$(45) \quad C \geq n, \quad C \geq \sum_{i=1}^{i=n} \frac{1}{d^{i-1}}.$$

We also define:

$$(46) \quad M(x) = \int_0^x CB |\log x|^{e_k-1} dx.$$

The point c is chosen so that:

$$(47) \quad 0 < c \leq b, \quad c \leq \frac{1}{\theta} \quad (\text{where } \log e = 1), \quad M(c) < 1.$$

This is the final determination of c to which we have referred on page 350; and this point c will, from now on, mark one end of the interval for x . Instead of $M(c)$, we shall, for the sake of brevity, write simply M .

The convergence of the series (34) will be proved by showing by mathematical induction that the following inequalities hold for all values of q :

* We might, for instance, take $B = \sum_{k=1}^{k=m} \sum_{l=1}^{l=e_k} \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} |b_{k,i}^{i,j}|$.

$$(48) \quad |\phi_{k,l,q}| \leq M^q \quad \left(\begin{matrix} k=1, 2, \dots, m \\ l=1, 2, \dots, e_k \end{matrix} \right).$$

For $q=0$ they are obviously true. Assuming them true for a special value of q , $q=q_1$, we will consider the cases outlined in (24) in turn.

I. From (27) we have:

$$(49) \quad |\phi_{k,l,q_1+1}| \leq$$

$$x^{R(r_k-r_s)} \sum_{t=1}^{t=l} \int_c^x \frac{1}{x} |dx| \dots \int_c^x \frac{1}{x} |dx| \int_c^x x^{-R(r_k-r_s)} B M^{q_1} |\log x|^{e_k-\lambda} |dx|$$

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after replacing

$$|\phi_{i,j,q_1}|, |\log x|^{h_{i,j}}, \sum \sum |b_{k,l}^{i,j}|$$

respectively by the greater values

$$M^{q_1}, |\log x|^{e_k-\lambda}, B$$

(Cf. (28), (30), (32)).

$$\begin{aligned} |\phi_{k,l,q_1+1}| &\leq M^{q_1} \sum_{t=1}^{t=l} \left[\frac{1}{R(r_k-r_s)} \right]^{t-1} \int_c^x B |\log x|^{e_k-\lambda} |dx| \text{ (Lemma IV.)} \\ &\leq M^{q_1} C \int_c^x B |\log x|^{e_k-\lambda} |dx| \leq M^{q_1} \int_c^x C B |\log x|^{e_k-\lambda} |dx| \\ &\leq M^{q_1} M(x) \leq M^{q_1+1}. \end{aligned}$$

From Lemma V. it follows that the limit of the right side of (49) is zero, when x approaches zero; for we have assumed that $B |\log x|^{e_k-\lambda}$ is integrable up to $x=0$ (cf. page 358), and therefore $B |\log x|^{e_k-\lambda}$ must also be integrable up to $x=0$. So (43) is verified for $q=q_1+1$.

II. From (29) we have for case (a):

$$\begin{aligned} (50) \quad |\phi_{k,l,q_1+1}| &\leq \sum_{t=1}^{t=l} \int_0^x \frac{1}{x} dx \dots \int_0^x \frac{1}{x} dx \int_0^x B M^{q_1} |\log x|^{e_k-\lambda} dx \\ &\quad \text{\textit{t} integrations} \\ &\leq M^{q_1} \sum_{t=1}^{t=l} \int_0^x B |\log x|^{e_k-\lambda+t-1} dx \quad \text{(Lemma II.)} \\ &\leq M^{q_1} \sum_{t=1}^{t=l} \int_0^x B |\log x|^{e_k-\lambda} dx \end{aligned}$$

$$\begin{aligned}
 (55) \quad & \frac{1}{|\log x|^{\iota-L}} \sum_{t=1}^{t=l-L} \int_c^x \frac{1}{x} |dx| \dots \int_c^x \frac{1}{x} |dx| \int_c^x B M^{q_1} |\log x|^{\epsilon_{\kappa-\lambda}} dx \\
 & \qquad \qquad \qquad t \text{ integrations} \\
 \leq & \sum_{t=1}^{t=l-L} |\log x|^{\iota-1} \left[\frac{M^{q_1}}{|\log x|^{\iota-L}} \int_0^c B |\log x|^{\epsilon_{\kappa-\lambda}} dx \right] \quad (\text{Lemma VII.}) \\
 \leq & M^{q_1} \sum_{t=1}^{t=l-L} |\log x|^{\iota-1-(\iota-L)} \int_0^c B |\log x|^{\epsilon_{\kappa-\lambda}} dx \\
 \leq & M^{q_1} \frac{(l-L)}{|\log x|} \int_c^c B |\log x|^{\epsilon_{\kappa-1}} dx \leq M^{q_1+1} \frac{(l-L)}{C}.
 \end{aligned}$$

Therefore :

$$|\phi_{k,l,q+1}| \leq M^{q+1} \left[\frac{L}{C} + \frac{l-L}{C} \right] = M^{q+1} \frac{l}{C} \leq M^{q+1}$$

The limit of the left side of (54) for $x = 0$ is zero by Lemma VI, while the same thing is true of the left side of (55) from the inequalities. Therefore (43) is true in this case for $q = q_1 + 1$.

From (33) we have for case (e):

$$(56) \quad |\phi_{\kappa, l, q_1+1}| \leq \frac{1}{|\log x|^{l-\lambda}} \left[\int_c^x \frac{1}{x} |dx| \dots \int_c^x \frac{1}{x} |dx| \int_c^x \frac{1}{x} M^{q_1} \frac{L}{G} M(x) |dx| \right. \\ \left. + \sum_{t=1}^{t=l-L} \int_c^x \frac{1}{x} |dx| \dots \int_c^x \frac{1}{x} |dx| \int_c^x B M^{q_1} \log x |e_{\kappa-\lambda}| |dx| \right].$$

In the first part of the bracket we have used the inequality obtained from (52):

$$|\log x|^{e_{\lambda}-e_{\kappa}} |\phi_{\kappa, L, q_1+1}| \leq M^q \frac{L}{C} M(x).$$

Now the only difference between the inequalities (53) and (56) is in the power of $|\log x|$ outside the brackets; and, since $|\log x|^{\ell-L} \geq |\log x|^{\ell-L}$, all the results that we have obtained from (53) will follow also from (56). * Then for all the sub-cases of II., (43) and (48) are true for $q = q_1 + 1$.

where the last transformation is valid since we are dealing with absolutely convergent series. Also:

$$(58) \quad \frac{1}{x} z_{k,l-1} = \sum_{q=0}^{q=\infty} \frac{1}{x} z_{k,l-1,q},$$

$$(59) \quad \frac{r_k}{x} z_{k,l} = \sum_{q=0}^{q=\infty} \frac{r_k}{x} z_{k,l,q}.$$

Adding (57), (58), (59), and changing slightly the summation on the right, we have:

$$(60) \quad \frac{1}{x} z_{k,l-1} + \frac{r_k}{x} z_{k,l} + \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j} \\ = \frac{1}{x} z_{k,l-1,0} + \frac{r_k}{x} z_{k,l,0} + \sum_{q=0}^{q=\infty} \left[\frac{1}{x} z_{k,l-1,q+1} + \frac{r_k}{x} z_{k,l,q+1} \right. \\ \left. + \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j,q} \right].$$

The series on the right is an absolutely and uniformly convergent series of continuous functions. If we replace the terms by their values given in (18) and (19), we can also write the series in the form:

$$(61) \quad \frac{d}{dx} z_{k,l,0} + \sum_{q=0}^{q=\infty} \frac{d}{dx} z_{k,l,q+1} = \sum_{q=0}^{q=\infty} \frac{d}{dx} z_{k,l,q},$$

which is the series of derivatives of the terms of (25). From this it follows that, if we differentiate the series for $z_{k,l}$ term by term, we shall obtain an absolutely and uniformly convergent series of continuous functions; and therefore $z_{k,l}$ has a continuous derivative at each point of our sub-interval, which is precisely (61), and this, as we have seen, is the same as the right side of (60). We can therefore write (60) in the form

$$\frac{d}{dx} z_{k,l} = \frac{1}{x} z_{k,l-1} + \frac{r_k}{x} z_{k,l} + \sum_{i=1}^{i=m} \sum_{j=1}^{j=e_i} b_{k,i}^{i,j} z_{i,j};$$

and now giving k and l all possible values we have precisely the system of equations (16). We have, then, the following result:

In any sub-interval of $0c$, not including the point $x = 0$, the functions $z_{k,l}$ represented by the series (25) are continuous in x , have continuous

first derivatives with regard to x , and satisfy the canonical system of equations at every point. The terms of the series for $z_{k,l}$ are given by the formulae (26), in which the functions $\phi_{k,l,q}$ are continuous in x throughout the whole of the interval $0b$, and reduce to zero when $x = 0$, except when $q = 0$. The functions $\phi_{k,l,0}$ are constants, zero in all cases but the following:

$$\phi_{k,l,0} = \frac{1}{(l-\lambda)!} \quad \lambda \leq l \leq e_k.$$

§ 5.

LINEAR INDEPENDENCE OF THE SOLUTIONS OF THE CANONICAL SYSTEM.

We have shown that corresponding to each elementary divisor $(r - r_k)^{e_k}$, there are e_k solutions of (16) obtained by giving λ the values $1, 2, \dots, e_k$; and for the development of these solutions we have required that $|y_{k,l}^{(j)}| |\log x|^{e_k-1}$ shall be integrable up to $x = 0$, where e_k is determined for the root r_k by the condition (23). We have, then, n solutions which may be written as follows:

$$(62) \quad \begin{aligned} z_{k,l}^{\kappa,\lambda} &= x^{r_\kappa} \phi_{k,l}^{\kappa,\lambda} & \left\{ \begin{array}{l} \text{when } Rr_k \neq Rr_\kappa \\ \text{or } Rr_k = Rr_\kappa, \quad k \neq \kappa \text{ and } l \leq L, \\ \text{or } Rr_k = Rr_\kappa, \quad k = \kappa \text{ and } l \leq \lambda, \end{array} \right. \\ z_{k,l}^{\kappa,\lambda} &= x^{r_\kappa} (\log x)^{l-L} \phi_{k,l}^{\kappa,\lambda} & Rr_k = Rr_\kappa, \quad k \neq \kappa \text{ and } L < l \leq e_k, \\ z_{k,l}^{\kappa,\lambda} &= x^{r_\kappa} (\log x)^{l-\lambda} \phi_{k,l}^{\kappa,\lambda} & \lambda < l \leq e_k. \end{aligned}$$

where the functions $\phi_{k,l}^{\kappa,\lambda}$ are continuous in x and

$$(63) \quad \begin{aligned} \phi_{k,l}^{\kappa,\lambda} \Big|_{x=0} &= 0 & \left\{ \begin{array}{l} k \neq \kappa \\ \text{or } k = \kappa, \text{ and } l < \lambda, \end{array} \right. \\ \phi_{k,l}^{\kappa,\lambda} \Big|_{x=0} &= \frac{1}{(l-\lambda)!} & \lambda \leq l \leq e, \\ (\kappa = 1, 2, \dots, m) & & (\lambda = 1, 2, \dots, e_\kappa). \end{aligned}$$

It is worth while to note three facts in regard to the z 's, which will be useful later on:

- I. $z_{k,l}^{\kappa,\lambda}$ does not involve $\log x$ explicitly whenever $\lambda \geq l$;
- (64) II. $\lim_{x=0} x^{-r_\kappa} z_{k,l}^{\kappa,\lambda} = 0$ when $Rr_\kappa > Rr_l$;
- III. $\lim_{x=0} x^{-r_\kappa} (\log x)^{-(e_\kappa-\lambda)} z_{k,l}^{\kappa,\lambda} = 0$ in all cases except the one, $k = \kappa$ and $l = e_\kappa$; and then the limit is $\frac{1}{(e_\kappa - \lambda)!}$.

We will now show that the n solutions (62) are linearly independent. Suppose they were not and that there were n constants $C_{\kappa, \lambda}$ not all zero such that:

$$(65) \quad \sum_{\kappa=1}^{\kappa=m} \sum_{\lambda=1}^{\lambda=e_{\kappa}} C_{\kappa, \lambda} z_{k, l}^{\kappa, \lambda} = 0 \quad \left(\begin{array}{l} k = 1, 2, \dots, m \\ l = 1, 2, \dots, e_k \end{array} \right).$$

It will be convenient to suppose that our notation is such that:

$$(66) \quad R r_1 \leq R r_2 \leq \dots \leq R r_m.$$

Consider first those equations of (65) for which $R r_k = R r_1$. We have:

$$(67) \quad \lim_{x \rightarrow 0} \sum_{\kappa=1}^{\kappa=m} \sum_{\lambda=1}^{\lambda=e_{\kappa}} C_{\kappa, \lambda} x^{-r_k} z_{k, l}^{\kappa, \lambda} = 0 \quad \begin{array}{l} R r_k = R r_1 \\ (l = 1, 2, \dots, e_k). \end{array}$$

Now let $l = 1$ in (67), and consider the limit of each term for any given value, within the range indicated, for k . For each term in which $R r_{\kappa} > R r_k = R r_1$ the limit is zero by II. We have left, now, only the terms:

$$(68) \quad C_{\kappa, \lambda} x^{-r_k} z_{k, 1}^{\kappa, \lambda} \quad R r_{\kappa} = R r_k = R r_1.$$

According to I. no logarithms appear explicitly in (68), and we can write:

$$(69) \quad C_{\kappa, \lambda} x^{-r_k} z_{k, 1}^{\kappa, \lambda} = C_{\kappa, \lambda} x^{r_{\kappa} - r_k} \phi_{k, 1}^{\kappa, \lambda}.$$

Now by (63) the limit of all such terms is zero except in the one case $\kappa = k$ and $\lambda = 1$, and for this term we have:

$$(70) \quad \lim_{x \rightarrow 0} C_{k, 1} x^{-r_k} z_{k, 1}^{k, 1} = \lim_{x \rightarrow 0} C_{k, 1} \phi_{k, 1}^{k, 1} = C_{k, 1}.$$

So in the case of $l = 1$ the limit (67) turns out to be $C_{k, 1}$ when we evaluate the limit term by term. Now this is impossible unless $C_{k, 1} = 0$; and so we must have, writing now κ in place of k :

$$(71) \quad C_{\kappa, 1} = 0 \quad R r_{\kappa} = R r_1.$$

Now consider in the same way the cases of (67) in which $l = 2$; and choose any one of the values of k indicated. Here again by II. the limit of each term is zero when $R r_{\kappa} > R r_k$; and we have left the terms:

$$(72) \quad C_{\kappa, \lambda} x^{-r_k} z_{k, 2}^{\kappa, \lambda} \quad \begin{array}{l} R r_{\kappa} = R r_k = R r_1 \\ \lambda \geq 2. \end{array}$$

Now by I. no logarithms appear explicitly in the terms (72), for the only case in which they could occur would be for $\lambda = 1$, but by (71) such terms do not appear. We can then write each term of (72)

$$(73) \quad C_{\kappa, \lambda} x^{-r_{\kappa}} z_{k,2}^{\kappa, \lambda} = C_{\kappa, \lambda} x^{r_{\kappa} - r_{\kappa}} \phi_{k,2}^{\kappa, \lambda};$$

and by (63) the limit of each is zero except in the one case:

$$(74) \quad \lim_{x \rightarrow 0} C_{k,2} x^{-r_k} z_{k,2}^{k,2} = \lim_{x \rightarrow 0} C_{k,2} \phi_{k,2}^{k,2} = C_{k,2}.$$

We have then, reasoning as before, and including the previous result,

$$(75) \quad C_{\kappa,1} = C_{\kappa,2} = 0 \quad R r_{\kappa} = R r_1.$$

Now the same reasoning can be applied until we have finally

$$(76) \quad C_{\kappa, \lambda} = 0; \quad R r_{\kappa} = R r_1, \\ \lambda \geq 1$$

and there are left in (65) only those solutions for which $R r_{\kappa} > R r_1$. Having thus disposed of all the r 's in (66) such that $R r_{\kappa} = R r_1$, we consider the next set of r 's whose real parts are all equal and as small as any other in the new set of r 's, and show, in exactly the same way, that the corresponding constants, $C_{\kappa, \lambda}$ are zero. Continuing in this way, we finally reach the result that all the constants in (65) are zero, and that the assumption made as to the dependence of the n solutions leads to a contradiction. *The n solutions are therefore linearly independent.*

§ 6

RETURN FROM THE CANONICAL SYSTEM TO THE ORIGINAL SYSTEM.

We will now return to the original system of equations (1). Each solution of the canonical system (16) determines a solution of the original system (1) on account of the relation (10). We have then the following n solutions of (1):

$$(77) \quad y_i^{\kappa, \lambda} = \sum_{k=1}^{k=m} \sum_{l=1}^{l=e_k} A_{i,k,l} z_{k,l}^{\kappa, \lambda} \quad \left(\begin{array}{l} i = 1, 2, \dots, n \\ \kappa = 1, 2, \dots, m \\ \lambda = 1, 2, \dots, e_{\kappa} \end{array} \right).$$

Now these n solutions are linearly independent, for the z solutions are linearly independent, and the determinant of the constants $A_{i,k,l}$ is not zero.

Now consider the following limit:

$$(78) \quad \lim_{x=0} x^{-r_\kappa} (\log x)^{-(e_\kappa-\lambda)} y_i^{\kappa, \lambda} \\ = \lim_{x=0} \sum_{k=1}^{k=e_\kappa} \sum_{l=1}^{l=e_k} A_{i, k, l} x^{-r_\kappa} (\log x)^{-(e_\kappa-\lambda)} z_{k, l}^{\kappa, \lambda}.$$

On the right, if we take the limit of each term in the summation, we shall find that it is zero in every case except the one in which $k = \kappa$ and $l = e_\kappa$ (cf. (64) III.). We have, then:

$$(79) \quad \lim_{x=0} x^{-r_\kappa} (\log x)^{-(e_\kappa-\lambda)} y_i^{\kappa, \lambda} = A_{i, \kappa, e_\kappa} \frac{1}{(e_\kappa - \lambda)!}.$$

So we can write as a set of solutions of (1), corresponding to the elementary divisors $(r - r_\kappa)^{e_\kappa}$:

$$(80) \quad y_i^{\kappa, \lambda} = x^{r_\kappa} (\log x)^{e_\kappa - \lambda} \psi_i^{\kappa, \lambda} \quad \begin{pmatrix} i = 1, 2, \dots, n \\ \lambda = 1, 2, \dots, e_\kappa \end{pmatrix},$$

where the functions $\psi_i^{\kappa, \lambda}$ are continuous in the interval $0 \leq x \leq c$, and such that:

$$(81) \quad \psi_i^{\kappa, \lambda}|_{x=0} = \frac{1}{(e_\kappa - \lambda)!} A_{i, \kappa, e_\kappa}.$$

The constants $A_{i, k, l}$ are determined independently of the functions $a_{i, j}$ in (1); and therefore all that we have said on page 346 in regard to certain sets of them as linearly independent solutions of the equations (4) in the special case of $a_{i, j} = 0$ holds equally well here.

In order to obtain solutions corresponding to $(r - r_\kappa)^{e_\kappa}$,* we assumed in the treatment of the canonical system that $|b_{k, l}^{i, j}| |\log x|^{e_\kappa - 1}$ was integrable up to $x = 0$. Now, since the coefficients $b_{k, l}^{i, j}$ are linear functions with constant coefficients of the coefficients $a_{i, j}$, it will be sufficient, in order to obtain solutions of (1) corresponding to $(r - r_\kappa)^{e_\kappa}$, to assume that $|a_{i, j}| |\log x|^{e_\kappa - 1}$ is integrable up to $x = 0$ for all values of i and j .

Our results may be stated as follows: *If $(r - r_\kappa)^{e_\kappa}$ is an elementary divisor of $\Delta(r)$, and if we consider all the elementary divisors $(r - r_k)^{e_k}$ of $\Delta(r)$ such that $Rr_k = Rr_\kappa$, and denote by e_κ that exponent which is as great as any other exponent in this set of elementary divisors, and assume that $|a_{i, j}| |\log x|^{e_\kappa - 1}$ is integrable up to $x = 0$; we can develop e_κ solutions of (1):*

* Cf. p. 349.

$$y_i^{\kappa, \lambda} = x^{r_\kappa} (\log x)^{e_\kappa - \lambda} \psi_i^{\kappa, \lambda}, \quad \begin{pmatrix} i = 1, 2, \dots, n \\ \lambda = 1, 2, \dots, e_\kappa \end{pmatrix}.$$

where $\psi_i^{\kappa, \lambda}$ is continuous in the neighborhood of $x = 0$ and

$$\psi_i^{\kappa, \lambda} \Big|_{x=0} = \frac{1}{(e_\kappa - \lambda)!} A_{i, \kappa, e_\kappa}.$$

If the root r_κ furnishes s elementary divisors :

$$(r - r_\kappa)^{e_\kappa}, (r - r_\kappa)^{e_\kappa + 1}, \dots, (r - r_\kappa)^{e_\kappa + s - 1},$$

then the constants :

$$A_{i, \kappa, e_\kappa}, A_{i, \kappa + 1, e_{\kappa + 1}}, \dots, A_{i, \kappa + s - 1, e_{\kappa + s - 1}} \quad (i = 1, 2, \dots, n)$$

are s linearly independent solutions of the equations (4) when $r = r_\kappa$.

The n solutions of the differential equations (1) that we obtain when $\kappa = 1, 2, \dots, m$ are linearly independent.

§ 7.

THE HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION OF THE n th ORDER.

We shall consider homogeneous linear differential equations which can be written as follows :

$$(82) \quad \frac{d^n y}{dx^n} + \left(\frac{\mu_1}{x} + p_1 \right) \frac{d^{n-1} y}{dx^{n-1}} + \left(\frac{\mu_2}{x} + p_2 \right) \frac{1}{x} \frac{d^{n-2} y}{dx^{n-2}} + \dots \\ + \left(\frac{\mu_n}{x} + p_n \right) \frac{1}{x^{n-1}} y = 0,$$

in which $\mu_1, \mu_2, \dots, \mu_n$ are constants, and p_1, p_2, \dots, p_n are functions of the real independent variable x , continuous in the interval $0 < x \leq b$, and such that their absolute values are integrable up to $x = 0$; in short, these p 's are to have the same properties as the functions $a_{i,j}$ in (1).

This equation can be replaced by a system of linear differential equations by the following substitutions :

$$(83) \quad x^i \frac{dy}{dx} = y_{n-i} \quad (i = 0, 1, \dots, n-1).$$

We thus get the system of differential equations:

$$(84) \quad \frac{dy_1}{dx} = - \left(\frac{\mu_1 - n + 1}{x} + p_1 \right) y_1 - \left(\frac{\mu_2}{x} + p_2 \right) y_2 - \dots - \left(\frac{\mu_n}{x} + p_n \right) y_n,$$

$$\frac{dy_i}{dx} = \frac{1}{x} y_{i-1} + \frac{n-i}{x} y_i \quad (i = 2, 3, \dots, n).$$

The characteristic determinant of (84) is:

$$(85) \quad \begin{vmatrix} r - n + 1 + \mu_1 & \mu_2 & \dots & \mu_{n-1} & \mu_n \\ -1 & r - n + 2 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & r - 1 & 0 \\ 0 & 0 & \dots & -1 & r \end{vmatrix}.$$

The adjoints of the first line are:

$$(86) \quad r(r-1) \dots (r-n+2); r(r-1) \dots (r-n+3); \dots$$

$$r(r-1); r; 1.$$

The characteristic equation is, then:

$$(87) \quad r(r-1) \dots (r-n+1) + \mu_1 r(r-1) \dots (r-n+2) + \dots$$

$$+ \mu_{n-2} r(r-1) + \mu_{n-1} r + \mu_n = 0.*$$

There is always one first minor of (85) which is not zero, the adjoint of μ_n ; and therefore if r_κ is a multiple root of (87), the only elementary divisor of (85) corresponding to r_κ is $(r - r_\kappa)^{e_\kappa}$, where e_κ is the multiplicity of the root r_κ .

We have seen that, corresponding to the elementary divisor $(r - r_\kappa)^{e_\kappa}$, there are e_κ linearly independent solutions of (84) of the form (80). Here the constants A_{i,κ,e_κ} must satisfy the set of equations:

$$(88) \quad (r_\kappa - n + 1 + \mu_1) A_{1,\kappa,e_\kappa} + \mu_2 A_{2,\kappa,e_\kappa} + \dots + \mu_n A_{n,\kappa,e_\kappa} = 0$$

$$- A_{i-1,\kappa,e_\kappa} + (r_\kappa - n + i) A_{i,\kappa,e_\kappa} = 0$$

$$(i = 2, 3, \dots, n).$$

This system of equations has essentially only one solution, namely:

$$(89) \quad A_{i,\kappa,e_\kappa} = \rho r_\kappa (r_\kappa - 1) \dots (r_\kappa - n + 1 + i) \quad (i = 1, 2, \dots, n-1),$$

$$A_{n,\kappa,e_\kappa} = \rho,$$

* (87) is also called the *indicial equation* of (82) for the point $x = 0$.

where ρ is a constant not zero. We may divide each solution by the corresponding number $\frac{\rho}{(e_\kappa - \lambda)!}$, and the resulting solutions may be written:

$$(90) \quad y_i^{\kappa, \lambda} = x^{r_\kappa} (\log x)^{e_\kappa - \lambda} E_i^{\kappa, \lambda} \quad \left(\begin{array}{l} i = 1, 2, \dots, n \\ \lambda = 1, 2, \dots, e_\kappa \end{array} \right),$$

where $E_i^{\kappa, \lambda} \Big|_{x=0} = r_\kappa (r_\kappa - 1) \dots (r_\kappa - n + 1 + i)$,

$$E_n^{\kappa, \lambda} \Big|_{x=0} = 1.$$

By means of (83) we can now return to solutions of the equation (82) with the following result:

If r_κ is a root of the characteristic equation, and $|p_i| |\log x|^{e_\kappa - 1}$ is integrable up to $x = 0$, where $e_\kappa \geq e_k$ for all k 's such that $R r_k = R r_\kappa$, e_k being the multiplicity of the root r_k , then the equation (82) has e_κ linearly independent solutions which may be written with their first $n - 1$ derivatives:

$$(91) \quad \begin{aligned} y^{\kappa, \lambda} &= x^{r_\kappa} (\log x)^{e_\kappa - \lambda} E_n^{\kappa, \lambda} \\ \frac{d^i y^{\kappa, \lambda}}{dx^i} &= x^{r_\kappa - i} (\log x)^{e_\kappa - \lambda} E_{n-i}^{\kappa, \lambda} \quad (\lambda = 1, 2, \dots, e_\kappa) \end{aligned}$$

where the functions $E_{n-i}^{\kappa, \lambda}$ are continuous in the neighborhood of $x = 0$ and,

$$(92) \quad \begin{aligned} E_n^{\kappa, \lambda} \Big|_{x=0} &= 1, \\ E_{n-i}^{\kappa, \lambda} \Big|_{x=0} &= r_\kappa (r_\kappa - 1) \dots (r_\kappa - i + 1). \end{aligned}$$

Even for the equation of the second order this theorem does not merely give the results of Professor Bôcher's paper above quoted, but goes a step farther, since in the case in which the two roots of the characteristic equation are equal, we require merely that:

$$\int_0^b (\log x) |p_1| dx, \quad \int_0^b (\log x) |p_2| dx$$

converge, while Professor Bôcher's method made it necessary for him to require that

$$\int_0^b (\log x)^2 |p_1| dx, \quad \int_0^b (\log x)^2 |p_2| dx$$

converge.*

* L. c. p. 48. The function $x q_1$ in this paper is the same as our p_2 .